A – 6

Numerical solution of KdV equation model of interfacial wave

L. H. Wiryanto, W. Djohan
Department of Mathematics,ITB, Indonesia
Email: leo@math.itb.ac.id, warsoma@math.itb.ac.id

Abstract

An implicit finite-difference method is developed to solve a Korteweg de Vries (KdV) equation, by constructing a system of equations that can be solved iteratively, so that the solution is stable. This method is then applied to a model of interfacial wave, so that we can study the effect of the physical parameters, such as the ratio of the fluid depths and of the fluid densities. In case the wave is solitary, the numerical solution can be compared to the analytical one. This is then used to observe the wave propagation of any incoming wave, such as a cosines wave.

1 Introduction

The Korteweg de Vries equation is a model of wave propagation in many physical phenomena, such as surface wave (original equation) derived by Kordeweg and de Vries [1] [1], wave generation derived by Shen et. al. [2], Wiryanto and Jamhuri [3] and interfacial wave derived by Segur and Hammack [4]. The equation is well-known describing the slow evolution of waves of small amplitude, that are long in comparison with the fluid depth.

For interfacial waves, Segur and Hammack [4] derived the KdV equation based on a two-fluid configuration, a layer of lighter fluid with depth h_1 and density ρ_1 overlies a heavier one with depth h_2 and density ρ_2, resting on a horizontal impermeable bed in a constant gravitational flied. Since the fluids are assumed to be ideal and the flow is irrotational, the problem is to determine the potential function for each layer. After scaling and using series expansion, they obtain the KdV equation for the interfacial elevation η as

$$c_0^{-1}\eta_t + \eta_x + \frac{3}{2}\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\eta\eta_x + \frac{1}{6}h_1h_2\eta_{xxx} = 0. \quad (1)$$

where

$$c_0^2 = \frac{g\triangle h_1 h_2}{h_1 + h_2}, \qquad \triangle = \frac{\rho_2 - \rho_1}{\rho_2}, \quad (2)$$

and g is the gravity acceleration. The aspect of KdV theory for long interfacial waves have been discussed by many researchers such as Keulegan [5], Long [6], Peter and Stoker [7], Benjamin [8], and Benney [9].

Kobota et. al. [10] derived the model of interfacial waves for the case in which the fluid is confined between two rigid walls and background density distribution is continues, which is different with the derivation in [4], using density discontinues. As the result, Kubota et. al. obtain the model containing an integral term in the principal-value sense, similar to the result obtained by Benjamin [11] and Ono [12]. Instead of integral term, Benjamin-Ono equation contains Hilbert transform. Tuck and Wiryanto [13] derived the Benjamin-Ono equation in different way, and compare the analytical solution of Benjamin-Ono equation with the numerical periodic solution. However, we will consider the numerical solution of (1). Segur and Hammack [4] presented the analytical solution, in solitary waves, and made the comparison to

 ISBN : 978-602-1037-00-3

their experiment. The numerical procedure of (1) requires initial and boundary conditions, and they pay an important role in performing the evolution of the interface waves. A numerical method is developed in obtaining the stable solution. Following Feng and Mitsui [14] and also Wiryanto and Achirul [15], an implicit finite difference method is used to construct a system of equations. We then solve the system of equation by Gauss-Seidel iteration, as the matrix is tridiagonal and diagonal dominant. The wave propagation is then performed for some initial waves.

2 Analytical Solution

The analytical solution of (1) can be obtained firstly by transforming the equation into the standard KdV equation. We define

$$\chi = \frac{x - c_0 t}{(h_1 h_2)^{1/2}}, \quad \tau = \frac{1}{6}\left(\frac{g\triangle}{h_1 + h_2}\right)^{1/2}, \quad f = \frac{3}{2}\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\eta, \tag{3}$$

so that the total derivative gives

$$\begin{aligned} \eta_t &= \frac{2}{3\left(\frac{1}{h_2} - \frac{1}{h_1}\right)}\left[f_\chi \frac{\partial \chi}{\partial t} + f_\tau \frac{\partial \tau}{\partial t}\right] \\ &= \frac{2}{3\left(\frac{1}{h_2} - \frac{1}{h_1}\right)}\left[-\frac{c_0}{(h_1 h_2)^{1/2}} f_\chi + \frac{1}{6}\left(\frac{g\triangle}{h_1 + h_2}\right)^{1/2} f_\tau\right], \end{aligned}$$

similarly for the other derivatives,

$$\eta_x = \frac{2}{3\left(\frac{1}{h_2} - \frac{1}{h_1}\right)(h_1 h_2)^{1/2}} f_\chi,$$

$$\eta_{xxx} = \frac{2}{3\left(\frac{1}{h_2} - \frac{1}{h_1}\right)(h_1 h_2)^{3/2}} f_{\chi\chi\chi}.$$

We then substitute those derivatives in (1), and simplify into

$$f_\tau + 6\alpha f f_\chi + \alpha f_{\chi\chi\chi} = 0, \tag{4}$$

where

$$\alpha = \frac{(h_1 + h_2)^{1/2}}{c_0^{-1} g^{1/2} \triangle^{1/2} (h_1 h_2)^{1/2}}.$$

From (2), the right hand side of α is 1, so that (4) is the standard KdV equation, and the solution is a solitary wave in form of

$$f(\chi, \tau) = 2K^2 \mathrm{sech}^2\left[K\left(\chi - 4K^2\tau - \chi_0\right)\right]. \tag{5}$$

K is constant related to the height crest, and χ_0 is the position of the crest. This analytical solution (5) is then used to obtain the solution of (1) by re-transforming the variables using (3), we have

$$\eta(x, t) = a^2\, \mathrm{sech}^2\left[p\left(x - x_0 - vt\right)\right], \tag{6}$$

where

$$p = \frac{a}{h_1 h_2}\sqrt{\frac{3\left(h_1 - h_2\right)}{4}}, \qquad v = \left(\frac{g\triangle h_1 h_2}{h_1 + h_2}\right)^{1/2}\left[1 + \frac{1}{2}\frac{h_1 - h_2}{h_1 h_2}a^2\right].$$

The solution (6) is solitary wave traveling with speed v, without changing the form of amplitude a^2 and width p. Larger p we have wave with thinner shape. This solution will be used as the reference to compare to our numerical procedure, before we apply to general initial condition of (1).

3 Numerical Procedure

A finite difference method is developed in calculating the solution of (1). The equation is first differentiated with respect to space x, to reduce the dispersive effect generated by the term containing third derivative. The equation that we solve is

$$c_0^{-1}\eta_{xt} + \eta_{xx} + \frac{3}{4}\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\left(\eta^2\right)_{xx} + \frac{1}{6}h_1 h_2 \eta_{xxxx} = 0. \qquad (7)$$

Meanwhile, the equation is linearized so that the discretized equation is unconditionally stable, see Feng and Mitsui [14]. We discetize the space x by denoting $x_j = jdx$, $j = 0, 1, 2, \cdots, J$, and the time t by denoting $t^n = ndt$, for $n = 0, 1, 2, \cdots$. dx and dt are small number as the length of space between two nodes and step time, respectively. From these discretisations, we then denote $\eta_j^n \approx \eta(x_j, t^n)$. The time derivative is approximated by forward difference, and the space x derivative is approximated by average central space, so that we have

$$\eta_{xt} \approx \frac{1}{dt}\left[\frac{\eta_{j+1}^{n+1} - \eta_{j-1}^{n+1}}{2dx} - \frac{\eta_{j+1}^{n} - \eta_{j-1}^{n}}{2dx}\right],$$

$$\eta_{xx} \approx \frac{1}{2}\left[\frac{\eta_{j+1}^{n+1} - 2\eta_j^{n+1} + \eta_{j-1}^{n+1}}{dx^2} + \frac{\eta_{j+1}^{n} - 2\eta_j^{n} + \eta_{j-1}^{n}}{dx^2}\right],$$

$$\eta_{xxxx} \approx \frac{1}{2}\left[\frac{\eta_{j+2}^{n+1} - 4\eta_{j+1}^{n+1} + 6\eta_j^{n+1} - 4\eta_{j-1}^{n+1} + \eta_{j-2}^{n+1}}{dx^4} + \frac{\eta_{j+2}^{n} - 4\eta_{j+1}^{n} + 6\eta_j^{n} - 4\eta_{j-1}^{n} + \eta_{j-2}^{n}}{dx^4}\right].$$

The non-linear term is approximated by

$$\left(\eta^2\right)_{xx} \approx \frac{\eta_{j+1}^{n}\eta_{j+1}^{n+1} - 2\eta_j^{n}\eta_j^{n+1} + \eta_{j-1}^{n}\eta_{j-1}^{n+1}}{dx^2}.$$

This is obtained by supposing $u = \eta^2$, and approximating

$$u_{xx} \approx \frac{1}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{dx^2} + \frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{dx^2}\right].$$

We then use Taylor series

$$u_j^{n+1} \approx u_j^n + u_\eta \eta_t|_j^n\, dt,$$

 ISBN : 978-602-1037-00-3

so that we have

$$\begin{aligned}
\frac{1}{2}\left[u_j^{n+1}+u_j^n\right] &= \frac{1}{2}\left[u_j^n + (u_\eta \eta_t)|_j^n\, dt + u_j^n\right] \\
&= \frac{1}{2}\left[2u_j^n + 2\eta\eta_t|_j^n\, dt\right] \\
&\approx \frac{1}{2}\left[2u_j^n + 2\eta_j^n\left(\frac{\eta_j^{n+1}-\eta_j^n}{dt}\right) dt\right] \\
&= \eta_j^n \eta_j^{n+1},
\end{aligned}$$

and we linearize $(\eta^2)_{xx}$ as written above.

For given η_j, $j = -2, -1, 0, \cdots, J+2$ at time step n, we calculate η_j at $n+1$ for $j = 0, 1, \cdots, J$ satisfying

$$a_2\eta_{j+2}^{n+1} + a_1\eta_{j+1}^{n+1} + a_0\eta_j^{n+1} + a_{-1}\eta_{j-1}^{n+1} + a_{-2}\eta_{j-2}^{n+1} = R, \tag{8}$$

where

$$\begin{aligned}
a_2 &= \frac{1}{12}\frac{c_0 h_1 h_2}{dx^4}, \\
a_1 &= \frac{1}{2dxdt} + \frac{c_0}{2dx^2} + \frac{3}{4}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\frac{c_0\eta_{j+1}^n}{dx^2} - \frac{1}{3}\frac{h_1 h_2 c_0}{dx^4}, \\
a_0 &= -\frac{c_0}{dx^2} - \frac{3}{2}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\frac{\eta_j^n c_0}{dx^2} + \frac{1}{2}\frac{h_1 h_2 c_0}{dx^4}, \\
a_{-1} &= -\frac{1}{2dxdt} + \frac{c_0}{2dx^2} + \frac{3}{4}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\frac{c_0\eta_{j-1}^n}{dx^2} - \frac{1}{3}\frac{h_1 h_2 c_0}{dx^4}, \\
a_{-2} &= \frac{1}{12}\frac{c_0 h_1 h_2}{dx^4},
\end{aligned}$$

and

$$R = \frac{1}{dt}\frac{\eta_{j+1}^n - \eta_{j-1}^n}{2dx} - \frac{c_0}{2}\frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{dx^2} - \frac{c_0 h_1 h_2}{12}\frac{\eta_{j+2}^n - 4\eta_{j+1}^n + 6\eta_j^n - 4\eta_{j-1}^n + \eta_{j-1}^n}{dx^4}.$$

The values of η_{-2}^{n+1}, η_{-1}^{n+1}, η_{J+1}^{n+1}, and η_{J+1}^{n+1} are given as the boundary condition. Since we observe that the wave propagates from left to right, η_{-2}^{n+1} and η_{-1}^{n+1} are given as input, and the other two can be determined by linearizing two values of η in the observation domain, physically it means that the right boundary is absorbed.

The system of equations constructed by (8) has coefficient matrix in form of penta-diagonal. To solve it, we can use Gauss-Seidel iteration. We stop the iteration when the maximum error of η^{n+1} between two two iterations is less than 10^{-7}. This result is then used for the next time step. The coordinates $(x_j, \eta_j^n)_{j=0}^J$ are plotted as the interfacial elevation at time t_n. The evolution of the interfacial wave can be observed by plotting for some time steps.

4 Result

The numerical procedure described in the previous section is used to solve (1) for some physical quantities, the upper fluid depth $h_1 = 1.5$ and density $\rho_1 = 0.7$ and lower fluid depth $h_2 = 1$

 ISBN : 978-602-1037-00-3

and density $\rho_2 = 1$. The lower fluid is chosen as the reference. The gravity acceleration is set $g = 10$. Most of our calculations uses $dx = 0.5$ and $dt = 0.125$. The number of points is $J = 1000$. As the initial condition, we follow the analytical solution (6)

$$\eta(x,0) = 0.04\,\mathrm{sech}^2\left[p\,(x-100)\right],$$

and similarly for the left boundaries for $\eta(x_{-1},t)$ and $\eta(x_{-2},t)$. Our calculation gives the coordinates of some points of the interfacial wave. We then plot those points for some times t_n by shifting upward for larger t_n as shown in Figure 1. The solitary wave is obtained without changing the form traveling to the right. This confirms to the analytical solution as described in the previous section in (6).

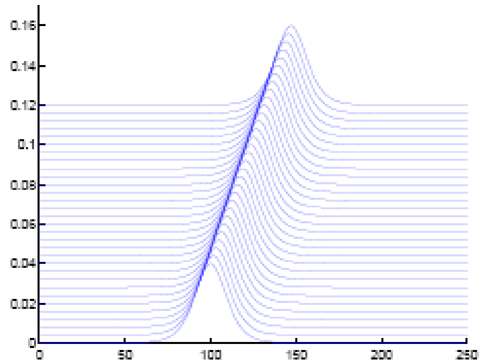
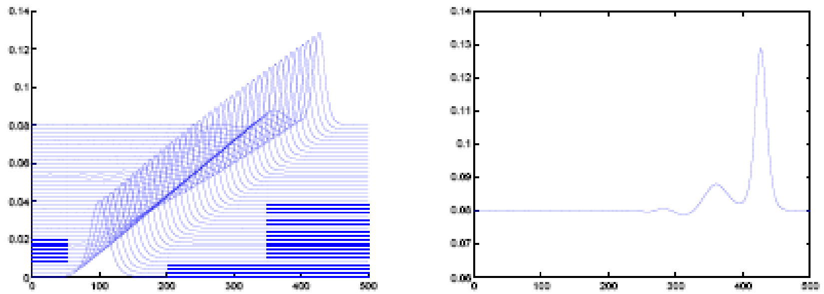


Figure 1: Plot of solitary interfacial wave, as the result of the initial condition $\eta(x,0) = 0.04\,\mathrm{sech}^2\left[p\,(x-100)\right]$



(a) (b)

Figure 2: (a). Plot of the wave evolution, calculated using the initial condition $\eta(x,0) = 0.04\,\mathrm{sech}^2\left[0.28\,(x-100)\right]$. (b). Plot of the numerical solution $\eta(x,5000)$.

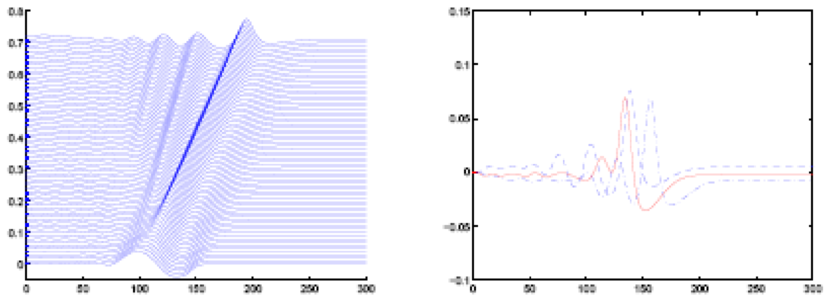
For various physical quantities of the upper and lower fluids, our calculations give similar plot to Fig 1, and they agree to the analytical solution (6) as long as the initial and boundary

 ISBN : 978-602-1037-00-3

conditions following that form. The effect of the physical quantities appears in p and v of (6). Now, we observe the solution of (1) for the initial condition

$$\eta(x,0) = 0.04\,\mathrm{sech}^2\left[0.28\,(x-100)\right],$$

the quantities related to the amplitude and the width of the wave do not satisfy as in the analytical solution (6). The wave evolution is shown in Figure 2a, the solitary form breaks up by appearing some small waves behind the main wave. To show that small waves more clearly, we plot $\eta(x,t)$ at the last time step $n = 40000$, equivalent to $t = 5000$, of our calculation in Figure 2b. The same initial condition is then used to calculate the interfacial wave for upper fluid density $\rho_1 = 0.3$. Some small waves appear behind the main wave, the difference with the previous result, for $\rho_1 = 0.7$, is the wave travels faster for smaller density. As the comparison, we calculate up to $n = 30000$, equivalent to $t = 3750$, the main wave almost reach the right observation boundary, at $x = 500$. Similarly for deeper upper fluid, the wave travels faster. This agrees to the analytical solution (6). The value v increases by increasing h_1 and decreasing ρ_1.



(a) (b)

Figure 3: (a). Plot of interfacial wave, as the result of the initial condition in form of sinusoidal function (b). Plot of some interfacial waves $\eta(x,625)$ for different physical quantities $\rho_1 = 0.1$ given in curve "-.", $h_1 = 3.0$ in curve "–" and the other is for $\rho_1 = 0.7$.

The numerical procedure described above is also used to calculate the wave evolution from the initial condition

$$\eta(x,0) = \begin{cases} 0.04\sin(0.26\pi x) & \text{for} \quad 75 < x < 150, \\ 0 & \text{for} \quad 0 < x < 75 \bigcup 150 < x < 500, \end{cases} \tag{9}$$

and we use zero for the left boundaries $\eta(x_0,t)$ and $\eta(x_{-1},t)$. The same physical quantities for the upper and lower fluids, as used in Figure 2, are used in that calculation. As the result, the wave propagates to the right and changes the form, such as presented in Figure 3(a). In the evolution, some small waves appear behind the main waves, similar to our calculations before, except the initial condition is (6) with the amplitude a^2 and the width p satisfying a special relation to the physical quantities involved in the equation (1). In Figure 3(b), we plot $\eta(x,625)$ in continues curve, from the evolution shown in Figure 3(a). We then recalculate to observe the effect of the physical quantities, especially for ρ_1 and h_1. First, we calculate using upper density $\rho_1 = 0.1$, and the same value for the other, i.e. $\rho_2 = 1$, $h_2 = 1$ and $h_1 = 1.5$, the initial sinusoidal (9) gives $\eta(x,625)$ as shown in Figure 3(b), plot in "-."curve. Similarly for $\rho_2 = 1$, $h_2 = 1$, $\rho_1 = 0.7$ and $h_1 = 3.0$, our calculation for $\eta(x,625)$ is plotted in "–"curve. From those plots, we can compare the wave evolution. The physical quantities effect to the wave speed and wave deformation.

5 Conclusion

We developed a numerical procedure based on a finite difference method for a Korteweg de Vries equation, the model for wave evolution at the interface between two fluids having different density. The numerical procedure gives solution that is in good agrement with the analytical solution, in form of solitary wave. Therefore, the code can be used to calculate for other types of waves, depending on the initial waves. In general, the model produces some other waves with smaller amplitude behind the main waves. The effect of the physical quantities is also observed, i.e. in producing smaller waves and the wave speed.

Acknowledgement 1 *The research of the problem was supported by Decentralization Research-Grant, Directorate General of Higher Education, Ministry of Education and Culture-Indonesia, contract numbers 1063b/I1.C01.2/PL/2014, and Research-Innovation Grant, Bandung Institute of Technology, contract number 1064w/I1.c01.2/PL/2014.*

References

[1] Korteweg, D. J., de Vries, G., On the change of form of long waves advancing in a rectangular channel, and on a new type of log stationary waves, *Phil. Mag.*, v39, 422-443, 1895.

[2] Shen, S.P., Shen, M.C., Sun, S.M., A model equation for steady surface wave over a bump, *J. Eng. Math*, v23, 315-323, 1989.

[3] Wiryanto, L.H., Jamhuri, M., Supercritical flow generating a solitary-like surface wave above a bump, *in preparation, 2014*.

[4] Segur, H., Hammack, J.L., Soliton models of long internal waves, *J. Fluid Mech.*, v118, 285-304, 1982.

[5] Keulegen, G.H., Characteristics of internal solitary waves, *J . Res. Nut. Bur. Stand.*, v51, 133-140, 1953.

[6] Long, R., Solitary waves in the one- and two-fluid systems, *Tellus* v8, 460-471, 1956. DOI: 10.1111/j.2153-3490.1956.tb01249.x

[7] Peters, S., Stoker, Solitary waves in liquids having non-constant density, *Commun. Pure Appl. Math.* v13, 115-164, 1960.

[8] Benjamin, T.B., Internal waves of finite amplitude and permanent form *J. Fluid Mech.* v25, 241-270, 1966. DOI: http://dx.doi.org/10.1017/S0022112066001630

[9] Benney, D.J., Long nonlinear waves in fluid flows, *J. Math. Phys.* v45, 52-63, 1966.

[10] Kubota, T., Ko, D.R.S., Dobbs, L.D., Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth, *J. Hydronautics* v12, 157-165, 1978.

[11] Benjamin, T.B., Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.*, v29, 559-599, 1967.

[12] Ono, H., Algebraic solitary waves in stratified fluid, *J. Phys. soc. Japan*, v39, 1082-1091, 1975.

[13] Tuck, E.O., Wiryanto, L.H., On steady periodic interfacial waves, *J. Eng. Math.*, v35, 71-84, 1999.

[14] Feng, B.-F., Mitsui, T., A finite difference method for the KdV and KP equations, *J. Comp. Appl. Math.*, v90, 95-116, 1998.

[15] Wiryanto, L.H., Achirul, A., An Implicit Finite Difference Methods for A Forced Kdv Equation *Jurnal Matematika* v11 11, 1-5, 2008.